Problem 1.12

Show that a substitution identical to that used in reduction of order $y(x) = u(x)f(x)$ can be used to eliminate the $y^{(n-1)}(x)$ term from an nth-order homogeneous linear differential equation. (When the one-derivative term has been eliminated from a linear second-order differential equation, the resulting equation is a *Schrödinger* equation.)

Solution

The general form of an nth-order homogeneous linear differential equation is the following.

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \tag{1}$$

Make the given substitution,

$$y(x) = u(x)f(x). \tag{2}$$

We now have to write the derivatives of y in terms of the new variable u. Take the first derivative.

$$y'(x) = u'(x)f(x) + u(x)f'(x) \tag{3}$$

Take the second derivative.

$$\begin{aligned} y''(x) &= u''(x)f(x) + u'(x)f'(x) + u'(x)f'(x) + u(x)f''(x) \\ &= u''(x)f(x) + 2u'(x)f'(x) + u(x)f''(x) \end{aligned} \tag{4}$$

Take the third derivative.

$$\begin{aligned} y'''(x) &= u'''(x)f(x) + u''(x)f'(x) + 2u''(x)f'(x) + 2u'(x)f''(x) + u'(x)f''(x) + u(x)f'''(x) \\ &= u'''(x)f(x) + 3u''(x)f'(x) + 3u'(x)f''(x) + u(x)f'''(x) \end{aligned}$$

From these derivatives we can observe a pattern and figure out what $y^{(n-1)}$ and $y^{(n)}$ are. The formula is very reminiscent to that of the binomial theorem.

$$y^{(n)}(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} u^{(n-k)}(x)f^{(k)}(x)$$

The term with the factorials gives us the numbers of Pascal's triangle. To get the formula for $y^{(n-1)}(x)$, just replace n with $n-1$ everywhere in the previous formula.

$$y^{(n-1)}(x) = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} u^{(n-1-k)}(x)f^{(k)}(x)$$

Finally, we can plug these expressions into equation (1).

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} u^{(n-k)}(x)f^{(k)}(x) + p_{n-1}(x) \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} u^{(n-1-k)}(x)f^{(k)}(x) \\ + \cdots + p_0(x)u(x)f(x) = 0$$

Since the new variable is u, we want $u^{(n-1)}$ to be eliminated now. The $u^{(n-1)}$ term is obtained from the first sum when $k = 1$ and from the second sum when $k = 0$. We don't care about the remaining terms, $u^{(n)}(x)$, $u^{(n-2)}(x)$, $u^{(n-3)}(x)$, and so on.

$$nu^{(n-1)}(x)f'(x) + p_{n-1}(x)u^{(n-1)}(x)f(x) + \cdots = 0$$

Factor out $u^{(n-1)}(x)$ from these two terms.

$$u^{(n-1)}(x)[nf'(x) + p_{n-1}(x)f(x)] + \cdots = 0$$

From this equation we can see that in order for the $u^{(n-1)}(x)$ term to disappear, we should choose $f(x)$ such that it satisfies

$$nf'(x) + p_{n-1}(x)f(x) = 0.$$

This is a first-order ODE that can be solved with separation of variables.

$$n\frac{df}{dx} = -p_{n-1}(x)f(x)$$

Separate variables.

$$\frac{df}{f} = -\frac{p_{n-1}(x)}{n}\,dx$$

Integrate both sides.

$$\ln|f| = -\frac{1}{n}\int^x p_{n-1}(s)\,ds + C$$

Exponentiate both sides.

$$|f| = e^{-\frac{1}{n}\int^x p_{n-1}(s)\,ds}e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign on the left side.

$$f(x) = \pm e^C e^{-\frac{1}{n}\int^x p_{n-1}(s)\,ds}$$

Use a new arbitrary constant A.

$$f(x) = Ae^{-\frac{1}{n}\int^x p_{n-1}(s)\,ds}$$

Therefore, the substitution,

$$y(x) = u(x)e^{-\frac{1}{n}\int^x p_{n-1}(s)\,ds},$$

can be used to eliminate the $y^{(n-1)}(x)$ term from an nth-order homogeneous linear differential equation.

Example: Schrödinger Equations

When $n = 2$, equation (1) reduces to a general second-order linear homogeneous ODE.

$$y'' + p_1(x)y' + p_0(x)y = 0$$

If we make the prescribed substitution,

$$y(x) = u(x)f(x),$$

then we can use equations (2) and (3) and (4) to write the ODE in terms of u.

$$u''(x)f(x) + 2u'(x)f'(x) + u(x)f''(x) + p_1(x)[u'(x)f(x) + u(x)f'(x)] + p_0(x)u(x)f(x)$$

Factor the left side like so.

$$u''(x)f(x) + u'(x)[2f'(x) + p_1(x)f(x)] + u(x)[f''(x) + p_1(x)f'(x) + p_0(x)f(x)] = 0$$

For $u'(x)$ to disappear, we require that $f(x)$ satisfies the ODE,

$$2f'(x) + p_1(x)f(x) = 0,$$

which is first-order and can be solved with separation of variables.

$$2\frac{df}{dx} = -p_1(x)f(x)$$

Separate variables.

$$\frac{df}{f} = -\frac{p_1(x)}{2}\,dx$$

Integrate both sides.

$$\ln|f| = -\frac{1}{2}\int^x p_1(s)\,ds + C$$

Exponentiate both sides.

$$|f| = e^{-\frac{1}{2}\int^x p_1(s)\,ds}e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign on the left side.

$$f(x) = \pm e^C e^{-\frac{1}{2}\int^x p_1(s)\,ds}$$

Use a new arbitrary constant A.

$$f(x) = Ae^{-\frac{1}{2}\int^x p_1(s)\,ds}$$

This is the function we should use for $f(x)$ in equation (2) to eliminate the first derivative in the ODE.